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On the existence of intelligent states associated with the non-compact group SU(1, 1)

G Vanden Berghe and H De Meyer†

Seminarie voor Wiskundige naturkunde, Rijksuniversiteit-Gent Krijgslaan 271-S9, B-9000 Gent, Belgium

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Abstract. The existence of quasi-intelligent states which are a generalisation of the intelligent states satisfying equality in the Heisenberg uncertainty relation is investigated for the hyperbolic angular momentum operators which generate the non-compact SU(1, 1) group.

1. Introduction

In quantum mechanics intelligent states associated with the Heisenberg algebra $\{x, p; [x, p] = \mathrm{i}\}$ are states which satisfy the Heisenberg equality $\Delta x \,.\, \Delta p = \frac{1}{2}$. Aragone *et al* (1974) were the first to construct the so called intelligent spin states. These satisfy the Heisenberg equality $\Delta J_1^2 \,.\, \Delta J_2^2 = \frac{1}{4}\langle J_3\rangle^2$ for the SU(2) angular momentum algebra, and are in fact the solutions of the linear equation

$$J'_\alpha|w\rangle \equiv (J_1 - \mathrm{i}\alpha J_2)|w\rangle = w|w\rangle, \tag{1.1}$$

where α is a real number. They can be obtained as linear combinations of Wigner states, but they can also be generated from the atomic coherent spin states or Bloch states (Aragone *et al* 1976). Recently Rashid (1977a) has developed a simple algebraic method which permits the reproduction of the intelligent spin states. Equation (1.1) is thereby solved for complex α with the aid of certain operator identities. The solutions obtained are called quasi-intelligent states. Only at the end of Rashid's paper is it verified that the states corresponding to real α-values are indeed intelligent. The great advantage of Rashid's method is that it becomes very easy to calculate the matrix elements of a wide variety of operators in the new basis. It is then also possible to develop a general method for the computation of the Clebsch–Gordan coefficients for the coupling of intelligent states (Rashid 1977b).

The non-compact group SU(1, 1) has been studied extensively in the past. Bargmann (1947) discussed all unitary irreducible representations (UIR) of that group and the corresponding matrix elements in the discrete basis where the compact generator is diagonal. Barut (1967) and Wybourne (1974) have given a unified treatment of the construction of the representations of eight three-parameter Lie algebras, including SU(1, 1). These papers show that the unitary representations of SU(1, 1) are all

† Aangesteld navorser bij het Nationaal Fonds voor Wetenschappelijk Onderzoek (Belgium).

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infinite dimensional. The eigenstates of the non-compact generators have been constructed with different techniques (Barut and Phillips 1968, Lindblad and Nagel 1970). The coherent states (the eigenstates of the lowering or raising operator) have been found in the discrete basis (Barut and Girardello 1971) as well as in the continuous basis (Hongoh 1976). It turns out that such eigenstates are in general not simple functions which can be normalised to unity, but in many cases they have to be interpreted as generalised functions.

In the present paper we investigate whether the concept of intelligent state can be enlarged to the non-compact SU(1, 1) group. Therefore we first review some properties of this group. Then an operator analogous to J'_α in (1.1) is constructed. An equation similar to (1.1) is solved for complex α-values, and the set of conditions to be satisfied such that the eigenstates become intelligent is derived and discussed.

2. Review of SU(1, 1)

The set of all matrices of the form

$$g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}, \qquad |\alpha|^2-|\beta|^2=1, \tag{2.1}$$

constitutes the non-compact group SU(1, 1). This group has three classes of conjugate one-parameter subgroups. A set of linearly independent elements of the Lie algebra of SU(1, 1) is deduced from the generators of the subgroups. They are found to satisfy

$$[K_1, K_2]=-\mathrm{i}K_3, \qquad [K_2, K_3]=\mathrm{i}K_1, \qquad [K_3, K_1]=\mathrm{i}K_2, \tag{2.2}$$

or in canonical form with $K_\pm=K_1\pm \mathrm{i}K_2$:

$$[K_3, K_\pm]=\pm K_\pm, \qquad [K_+, K_-]=-2K_3, \tag{2.3}$$

$K_\pm$ being the ladder operators. Using a coupled boson representation Schwinger (1965) showed that K_+ permits the construction of the Wigner state $|j+1\ m\rangle$ from the state $|j\,m\rangle$. The operators K_1, K_2, K_3 are then called the hyperbolic angular momentum operators. The Casimir invariant is given by

$$C_2=K_3^2-K_1^2-K_2^2. \tag{2.4}$$

The unitary irreducible representations can be grouped into three classes according to the spectrum of C_2 and K_3. In all cases a standard basis $\{|\Phi\ m\rangle\}$ can be chosen with:

$$\begin{aligned}
&\langle\Phi\ m|\Phi\ m'\rangle=\delta_{mm'},\\
&C_2|\Phi\ m\rangle=\Phi(\Phi+1)|\Phi\ m\rangle,\\
&K_3|\Phi\ m\rangle=m|\Phi\ m\rangle,\\
&K_+|\Phi\ m\rangle=[(m+\Phi+1)(m-\Phi)]^{1/2}|\Phi\ m+1\rangle,\\
&K_-|\Phi\ m\rangle=[(m+\Phi)(m-\Phi-1)]^{1/2}|\Phi\ m-1\rangle.
\end{aligned} \tag{2.5}$$

The three series of UIR are:

(i) the continuous principal series

$$\Phi=-\tfrac{1}{2}+\mathrm{i}s \qquad 0<s<\infty \qquad \begin{cases} m=0, \pm1, \pm2, \ldots & C_\Phi^0,\\ m=\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots & C_\Phi^1,\end{cases}$$

(ii) the supplementary series

$$-\tfrac{1}{2}<\Phi<0 \qquad m=0, \pm 1, \pm 2, \ldots \qquad E_{\Phi},$$

(iii) the discrete principal series

$$\Phi=-\tfrac{1}{2},-1,-\tfrac{3}{2}, \ldots \qquad \begin{cases} m=-\Phi,-\Phi+1, \ldots & D_{\Phi}^{+}, \\ m=\Phi, \Phi-1, \ldots & D_{\Phi}^{-}. \end{cases}$$

In what follows we shall only refer to the D_{Φ}^{+} representation. The non-compact generators K_1, K_2 have a continuous spectrum.

A representation of the Lie algebra (2.2) in terms of boson creation and annihilation operators, which is suitable for the diagonalisation of the compact generator K_3, is given by Barut and Phillips (1968):

$$K_1=\tfrac{1}{2}\mathrm{i}(a_1^* a_2+a_2^* a_1), \qquad K_2=\tfrac{1}{2}(a_1^* a_2-a_2^* a_1), \qquad K_3=\tfrac{1}{2}(a_1^* a_1-a_2^* a_2), \tag{2.6}$$

or

$$K_{+}=\mathrm{i} a_1^* a_2, \qquad K_{-}=\mathrm{i} a_2^* a_1. \tag{2.7}$$

In terms of these boson operators $|\Phi\, m\rangle$ can be written as

$$|\Phi\, m\rangle=A_m(a_1^*)^{\Phi+m}(a_2^*)^{\Phi-m}. \tag{2.8}$$

The normalisation constant A_m can be determined up to a phase factor by the requirement of unitarity (Barut and Phillips 1968), and one finds:

$$A_m=\mathrm{e}^{\mathrm{i}\tau(m)}\left|\frac{(-1)^{\Phi+m}\Gamma(m-\Phi)}{\Gamma(\Phi+m+1)\Gamma(-2\Phi)}\right|^{1/2}, \qquad \tau(m)\in\mathbb{R}. \tag{2.9}$$

3. Quasi-intelligent states

Following Merzbacher (1970) and Rashid (1977a) a quasi-intelligent state associated with the commutator $[A, B]=\mathrm{i}C$, where A, B, C are Hermitian operators, is an eigenstate of the operator $(A-\mathrm{i}\alpha B)/(1-\alpha^2)^{1/2}$. Herein α is a complex constant different from 1 and -1. For real α-values the corresponding states are intelligent in the sense that the Heisenberg equality $(\Delta A)^2(\Delta B)^2=\frac{1}{4}|\langle C\rangle|^2$ is satisfied.

Taking into account the commutators (2.2) for the Hermitian generators K_1, K_2 and K_3 we define the following operators:

$$K_3'(\theta)=\cosh\theta\, K_2+\mathrm{i}\sinh\theta\, K_1, \tag{3.1}$$

$$K_{\pm}'(\theta)=K_3\pm\cosh\theta\, K_1\mp\mathrm{i}\sinh\theta\, K_2. \tag{3.2}$$

The complex constant θ, related to α by

$$\cosh\theta=\frac{1}{(1-\alpha^2)^{1/2}}, \qquad \sinh\theta=-\frac{\alpha}{(1-\alpha^2)^{1/2}},$$

is introduced in order to avoid denominators in the definitions. K_3' is a linear combination of two non-compact generators and its spectrum is continuous. Therefore, the quasi-intelligent states are the solutions of the equation

$$K_3'(\theta)|\Phi\,\phi\,\theta\rangle=\phi|\Phi\,\phi\,\theta\rangle, \tag{3.3}$$

and they are labelled by the value of the complex constant ϕ and by the eigenvalues of the commuting operators K'_3 and C_2. It should be noted that K'_3 is only Hermitian when θ is pure imaginary. It follows from (3.1) and (3.2) that K'_+ and K'_- are ladder operators changing $|\phi\rangle$ to $|\phi+\mathrm{i}\rangle$ and $|\phi-\mathrm{i}\rangle$ respectively. In the following we shall often restrict ϕ to take real values only, with the consequence that the matrix elements $\langle\Phi\,\phi'\,\theta|K'_\pm(\theta)|\Phi\,\phi\,\theta\rangle$ cannot be given a sense even as distributions in ϕ and ϕ' (Lindblad and Nagel 1970).

We now solve equation (3.3) for $\mathrm{Re}(\theta)=0$. Therefore we introduce new boson operators by

$$d_1=(\mathrm{i}\,\mathrm{e}^{\theta/2}a_1+\mathrm{e}^{-\theta/2}a_2)/\sqrt{2},\qquad d_2=(\mathrm{e}^{\theta/2}a_1+\mathrm{i}\,\mathrm{e}^{-\theta/2}a_2)/\sqrt{2},$$

or

$$a_1=\mathrm{e}^{-\theta/2}(-\mathrm{i}d_1+d_2)/\sqrt{2},\qquad a_2=\mathrm{e}^{\theta/2}(d_1-\mathrm{i}d_2)/\sqrt{2},\tag{3.5}$$

so that $d_i=\partial/\partial d_i^*$ and $[d_i,d_j^*]=\delta_{ij}$. In terms of these operators K'_3 can be written as

$$K'_3(\theta)=\tfrac{1}{2}\mathrm{i}(d_1^*d_1-d_2^*d_2),\tag{3.6}$$

and consequently a solution $|\Phi\,\phi\,\theta\rangle$ of (3.3) is formally given by

$$|\Phi\,\phi\,\theta\rangle=A_\phi(d_1^*)^{\Phi-\mathrm{i}\phi}(d_2^*)^{\Phi+\mathrm{i}\phi},\tag{3.7}$$

whereby A_ϕ is a normalisation factor to be determined later. With the insertion of (3.5) the solution (3.7) becomes

$$|\Phi\,\phi\,\theta\rangle=A_\phi\left(\frac{-\mathrm{i}\,\mathrm{e}^{-\theta/2}a_1^*+\mathrm{e}^{\theta/2}a_2^*}{\sqrt{2}}\right)^{\Phi-\mathrm{i}\phi}\left(\frac{\mathrm{e}^{-\theta/2}a_1^*-\mathrm{i}\,\mathrm{e}^{\theta/2}a_2^*}{\sqrt{2}}\right)^{\Phi+\mathrm{i}\phi}$$

which by use of the binomial theorem reduces to:

$$|\Phi\,\phi\,\theta\rangle=\frac{A_\phi}{2^\Phi}\sum_{N,N'}\binom{\Phi+\mathrm{i}\phi}{N}\binom{\Phi-\mathrm{i}\phi}{N'}(a_1^*)^{N+N'}(a_2^*)^{2\Phi-N-N'}$$
$$\times(-\mathrm{i}\,\mathrm{e}^{-\theta/2})^{N'}(\mathrm{e}^{\theta/2})^{\Phi-\mathrm{i}\phi-N'}(\mathrm{e}^{-\theta/2})^{N}(-\mathrm{i}\,\mathrm{e}^{\theta/2})^{\Phi+\mathrm{i}\phi-N}.$$

Letting $N+N'=\Phi+m$ one finds

$$|\Phi\,\phi\,\theta\rangle=\frac{A_\phi}{2^\Phi}\sum_m\frac{\mathrm{e}^{-m\theta}}{A_m}|m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m}\sum_{N'}\binom{\Phi+\mathrm{i}\phi}{\Phi-N'+m}\binom{\Phi-\mathrm{i}\phi}{N'}(-1)^{N'},\tag{3.8}$$

whereby (2.8) has been used. Introducing the hypergeometric function, one obtains:

$$|\Phi\,\phi\,\theta\rangle=\frac{A_\phi}{2^\Phi}\sum_m\frac{\mathrm{e}^{-m\theta}}{A_m}|m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m}\frac{\Gamma(\Phi+\mathrm{i}\phi+1)}{\Gamma(\mathrm{i}\phi-m+1)\Gamma(\Phi+m+1)}$$
$$\times{}_2F_1(-\Phi-m,-\Phi+\mathrm{i}\phi;\mathrm{i}\phi-m+1;-1).\tag{3.9}$$

The normalisation constant A_ϕ follows again from the unitary requirement and is formally the same as the normalisation constant for the eigenstates of the K_2 operator given by Barut and Phillips (1968):

$$|A_\phi|^2=\frac{1}{2\pi}\frac{\Gamma(-\Phi+\mathrm{i}\phi)\Gamma(-\Phi-\mathrm{i}\phi)}{\Gamma(-2\Phi)}.\tag{3.10}$$

Finally we want to remark that one has to take great care in handling complex powers of complex functions or constants. In order to avoid ambiguities we adopt the convention that in

$$z^u = e^{z \ln u} \qquad u, z \in \mathbb{C},$$

the logarithm takes its principal value ($-\pi < \arg \ln u \leq \pi$). In doing so the ambiguities that arise by multiplying d_1 and d_2 defined in (3.5) by independent phase factors (leaving (3.6) invariant) disappear.

4. Solutions for complex θ

Although the method of § 3 is only applicable for $\mathrm{Re}(\theta) = 0$ it is reasonable to believe that (3.9) is valid for any complex value of θ. In either case however the solution $|\Phi\ \phi\ \theta\rangle$ is not completely determined since by virtue of (2.9) an m-dependent phase associated with A_m can subsist in the m-summation. We shall further specify $\tau(m)$ by direct calculation. Using (2.5), (3.1) and (3.8) one finds:

$$\begin{aligned}
K_3'(\theta)|\Phi\ \phi\ \theta\rangle &= \tfrac{1}{2}\mathrm{i}(e^{\theta}K_- - e^{-\theta}K_+)|\Phi\ \phi\ \theta\rangle \\
&= \frac{\mathrm{i}}{2}\frac{A_\phi}{2^\Phi}\sum_m \frac{e^{-m\theta}}{A_{m+1}}|\Phi\ m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m+1}\sum_N(-1)^N\binom{\Phi-\mathrm{i}\phi}{N}\binom{\Phi+\mathrm{i}\phi}{\Phi-N+m+1} \\
&\quad\times[(m+\Phi+1)(m-\Phi)]^{1/2} - \frac{\mathrm{i}}{2}\frac{A_\phi}{2^\Phi}\sum_m \frac{e^{-m\theta}}{A_{m-1}}|\Phi\ m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m-1} \\
&\quad\times\sum_N(-1)^N\binom{\Phi-\mathrm{i}\phi}{N}\binom{\Phi+\mathrm{i}\phi}{\Phi-N+m-1}[(m+\Phi)(m-\Phi-1)]^{1/2},
\end{aligned}$$

where m has been changed to $m+1$ in the first sum and to $m-1$ in the second. With the help of (2.9) one thus obtains:

$$\begin{aligned}
K_3'(\theta)|\Phi\ \phi\ \theta\rangle &= -\frac{1}{2}\frac{A_\phi}{2^\Phi}\sum_m \frac{e^{-m\theta}}{A_m}|\Phi\ m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m} \\
&\quad\times\Bigg[\exp[\mathrm{i}(\tau(m)-\tau(m+1))](\Phi+m+1)\sum_N(-1)^N\binom{\Phi-\mathrm{i}\phi}{N}\binom{\Phi+\mathrm{i}\phi}{\Phi-N+m+1} \\
&\quad+\exp[\mathrm{i}(\tau(m)-\tau(m-1))](m-\Phi-1)\sum_N(-1)^N\binom{\Phi-\mathrm{i}\phi}{N}\binom{\Phi+\mathrm{i}\phi}{\Phi-N+m-1}\Bigg] \\
&= -\frac{1}{2}\frac{A_\phi}{2^\Phi}\sum_m \frac{e^{-m\theta}}{A_m}|\Phi\ m\rangle(-\mathrm{i})^{\mathrm{i}\phi-m}\frac{\Gamma(\Phi+\mathrm{i}\phi+1)}{\Gamma(\mathrm{i}\phi-m+1)\Gamma(\Phi+m+1)} \\
&\quad\times\Big((\mathrm{i}\phi-m)\exp[\mathrm{i}(\tau(m)-\tau(m+1))] \\
&\quad\times{}_2F_1(-\Phi-m-1, -\Phi+\mathrm{i}\phi; \mathrm{i}\phi-m; -1) \\
&\quad+\frac{(m-\Phi-1)(\Phi+m)}{\mathrm{i}\phi-m+1}\exp[\mathrm{i}(\tau(m)-\tau(m-1))] \\
&\quad\times{}_2F_1(-\Phi-m+1, -\Phi+\mathrm{i}\phi; \mathrm{i}\phi-m+2; -1)\Big).
\end{aligned} \tag{4.1}$$

By adaptation of a formula for hypergeometric functions given by Vilenkin (1969, p 383):

$$(a+1-b-c)\,{}_2F_1(a,b;c;-1)$$
$$=-(c-1)\,{}_2F_1(a-1,b;c-1;-1)+\frac{a(c-b)}{c}\,{}_2F_1(a+1,b;c+1;-1)$$

into the appropriate form

$$-(\mathrm{i}\phi-m)\,{}_2F_1(-\Phi-m-1,-\Phi+\mathrm{i}\phi;\mathrm{i}\phi-m;-1)$$
$$+\frac{(\Phi+m)(m-\Phi-1)}{\mathrm{i}\phi-m+1}\,{}_2F_1(-\Phi-m+1,-\Phi+\mathrm{i}\phi;\mathrm{i}\phi-m+2;-1)$$
$$=-2\mathrm{i}\phi\,{}_2F_1(-\Phi-m,-\Phi+\mathrm{i}\phi;\mathrm{i}\phi-m+1;-1), \tag{4.2}$$

one recognises from (4.1) and (4.2) that (3.9) is a solution of (3.3) if one makes the choice:

$$\exp[\mathrm{i}(\tau(m)-\tau(m+1))]=\mathrm{i} \qquad \text{and} \qquad \exp[\mathrm{i}(\tau(m)-\tau(m-1))]=-\mathrm{i}, \qquad 2m\in\mathbb{Z}.$$

These equations which are equivalent, are satisfied for

$$\tau(m)=-\tfrac{1}{2}m\pi+\tau,$$

τ being an arbitrary m-independent constant which can without loss of generality be set equal to zero now. It follows that

$$A_m=(-\mathrm{i})^m\left|\frac{\Gamma(m-\Phi)}{\Gamma(\Phi+m+1)\Gamma(-2\Phi)}\right|^{1/2}, \tag{4.3}$$

and taking into account the expression (4.3) for A_m, (3.9) is indeed valid for any θ-value.

It may appear striking that $|\Phi\;\phi\;\theta\rangle$ only differs by factors $\mathrm{e}^{-m\theta}$ from an eigenstate of the non-compact generator K_2 given by Barut and Phillips (1968):

$$K_2|\Phi\;\lambda\rangle=\lambda|\Phi\;\lambda\rangle,$$
$$|\Phi\;\lambda\rangle=\frac{A_\lambda}{2^\Phi}\sum_m\frac{|\Phi\;m\rangle}{A_m}(\mathrm{i})^{2\Phi-\mathrm{i}\lambda+m}\frac{\Gamma(\Phi+\mathrm{i}\lambda+1)}{\Gamma(\mathrm{i}\lambda-m+1)\Gamma(\Phi+m+1)}$$
$$\times{}_2F_1(-\Phi-m,-\Phi+\mathrm{i}\lambda;\mathrm{i}\lambda-m+1;-1). \tag{4.4}$$

Nevertheless this could also have been predicted as follows. With the aid of the commutation relations (2.2) one easily proves that

$$\mathrm{e}^{-\theta K_3}K_2\,\mathrm{e}^{\theta K_3}=\cosh\theta\,K_2+\mathrm{i}\sinh\theta\,K_1=K_3'(\theta) \qquad \theta\in\mathbb{C}.$$

So one obtains also that

$$K_3'(\theta)\,\mathrm{e}^{-\theta K_3}=\mathrm{e}^{-\theta K_3}K_2, \tag{4.5}$$

showing that if $|\Phi\;\lambda\rangle$ is an eigenstate of K_2, the state $\mathrm{e}^{-\theta K_3}|\Phi\;\lambda\rangle$ is an eigenstate of $K_3'(\theta)$ with the same eigenvalue.

Finally it should be noted that (3.9) could also have been obtained with the method of Lindblad and Nagel (1970). Hereby the difference equation resulting from

$$\tfrac{1}{2}\mathrm{i}(\mathrm{e}^{\theta}K_- - \mathrm{e}^{-\theta}K_+)|\Phi\ \phi\ \theta\rangle = \phi|\Phi\ \phi\ \theta\rangle$$

is solved with given initial conditions by means of Laplace's method.

5. Intelligent states

We now examine the conditions that have to be imposed such that a state of the form (3.9) becomes intelligent. From (3.3), (3.1) it follows immediately that

$$\langle\Phi\ \phi_2\ \theta_2|\cosh\theta_1\ K_2 + \mathrm{i}\sinh\theta_1\ K_1|\Phi\ \phi_1\ \theta_1\rangle = \phi_1\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle. \tag{5.1}$$

We have set $\Phi_1 = \Phi_2 = \Phi$ since all matrix elements vanish for $\Phi_1 \neq \Phi_2$. Since

$$K_3'(\theta_2)|\Phi\ \phi_2\ \theta_2\rangle = \phi_2|\Phi\ \phi_2\ \theta_2\rangle$$

and ϕ_2 is restricted to be a real number, one obtains also

$$\langle\Phi\ \phi_2\ \theta_2|[K_3'(\theta_2)]^{\dagger} = \langle\Phi\ \phi_2\ \theta_2|\phi_2,$$

or

$$\langle\Phi\ \phi_2\ \theta_2|\cosh\theta_2^* K_2 - \mathrm{i}\sinh\theta_2^* K_1|\Phi\ \phi_1\ \theta_1\rangle = \phi_2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle. \tag{5.2}$$

Combining (5.1) and (5.2) one finds the matrix elements of K_1 and K_2:

$$\mathrm{i}\sinh(\theta_1+\theta_2^*)\langle\Phi\ \phi_2\ \theta_2|K_1|\Phi\ \phi_1\ \theta_1\rangle = (\phi_1\cosh\theta_2^* - \phi_2\cosh\theta_1)\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle, \tag{5.3}$$

$$\mathrm{i}\sinh(\theta_1+\theta_2^*)\langle\Phi\ \phi_2\ \theta_2|K_2|\Phi\ \phi_1\ \theta_1\rangle = (\phi_1\sinh\theta_2^* + \phi_2\sinh\theta_1)\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_2\ \theta_1\rangle. \tag{5.4}$$

In an analogous way one proves that

$$\langle\Phi\ \phi_2\ \theta_2|(\cosh\theta_1\ K_2 + \mathrm{i}\sinh\theta_1\ K_1)^2|\Phi\ \phi_1\ \theta_1\rangle = \phi_1^2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle, \tag{5.5}$$

$$\langle\Phi\ \phi_2\ \theta_2|(\cosh\theta_2^*\ K_2 - \mathrm{i}\sinh\theta_2^*\ K_1)^2|\Phi\ \phi_1\ \theta_1\rangle = \phi_2^2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle, \tag{5.6}$$

$$\begin{aligned}\langle\Phi\ \phi_2\ \theta_2|&(\cosh\theta_2^*\ \mathrm{K}_2 - \mathrm{i}\sinh\theta_2^*\ K_1)(\cosh\theta_1\ K_2 + \mathrm{i}\sinh\theta_1\ K_1)|\Phi\ \phi_1\ \theta_1\rangle\\ &= \phi_1\ \phi_2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle,\end{aligned} \tag{5.7}$$

and by suitable combination of (5.5)–(5.7) the matrix elements of K_1^2 and K_2^2 can be derived with the help of the commutation relations (2.2):

$$\begin{aligned}\sinh^2(\theta_1+\theta_2^*)&\langle\Phi\ \phi_2\ \theta_2|K_1^2|\Phi\ \phi_1\ \theta_1\rangle\\ &= -(\phi_1\cosh\theta_2^* - \phi_2\cosh\theta_1)^2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle\\ &\quad+\sinh(\theta_1+\theta_2^*)\cosh\theta_1\cosh\theta_2^*\langle\Phi\ \phi_2\ \theta_2|K_3|\Phi\ \phi_1\ \theta_1\rangle,\end{aligned} \tag{5.8}$$

$$\begin{aligned}\sinh^2(\theta_1+\theta_2^*)&\langle\Phi\ \phi_2\ \theta_2|K_2^2|\Phi\ \phi_1\ \theta_1\rangle\\ &= (\phi_1\sinh\theta_2^* + \phi_2\sinh\theta_1)^2\langle\Phi\ \phi_2\ \theta_2|\Phi\ \phi_1\ \theta_1\rangle\\ &\quad+\sinh(\theta_1+\theta_2^*)\sinh\theta_1\sinh\theta_2^*\langle\Phi\ \phi_2\ \theta_2|K_3|\Phi\ \phi_1\ \theta_1\rangle.\end{aligned} \tag{5.9}$$

Defining as usual

$$\langle\Delta K_i^2\rangle = \langle K_i^2\rangle - \langle K_i\rangle^2 \qquad i = 1, 2, \tag{5.10}$$

one obtains immediately from (5.3), (5.4), (5.8) and (5.9):

$$\begin{aligned}\sinh^2(\theta_1+\theta_2^*)\langle\Phi\,\phi_2\,\theta_2|\Delta K_1^2|\Phi\,\phi_1\,\theta_1\rangle &= [(\phi_1\cosh\theta_2^*-\phi_2\cosh\theta_1)\langle\Phi\,\phi_2\,\theta_2|\Phi\,\phi_1\,\theta_1\rangle]^2\\ &\quad-(\phi_1\cosh\theta_2^*-\phi_2\cosh\theta_1)^2\langle\Phi\,\phi_2\,\theta_2|\Phi\,\phi_1\,\theta_1\rangle\\ &\quad+\sinh(\theta_1+\theta_2^*)\cosh\theta_1\cosh\theta_2^*\langle\Phi\,\phi_2\,\theta_2|K_3|\Phi\,\phi_1\,\theta_1\rangle, \end{aligned}\tag{5.11}$$

$$\begin{aligned}\sinh^2(\theta_1+\theta_2^*)\langle\Phi\,\phi_2\,\theta_2|\Delta K_2^2|\Phi\,\phi_1\,\theta_1\rangle &= -[(\phi_1\sinh\theta_2^*+\phi_2\sinh\theta_1)\langle\Phi\,\phi_2\,\theta_2|\Phi\,\phi_1\,\theta_1\rangle]^2\\ &\quad+(\phi_1\sinh\theta_2^*+\phi_2\sinh\theta_1)^2\langle\Phi\,\phi_2\,\theta_2|\Phi\,\phi_1\,\theta_1\rangle\\ &\quad+\sinh(\theta_1+\theta_2^*)\sinh\theta_1\sinh\theta_2^*\langle\Phi\,\phi_2\,\theta_2|K_3|\Phi\,\phi_1\,\theta_1\rangle. \end{aligned}\tag{5.12}$$

Considerable simplifications occur in (5.11), (5.12) if

$$\langle\Phi\,\phi_2\,\theta_2|\Phi\,\phi_1\,\theta_1\rangle = 0 \text{ or } 1,\tag{5.13}$$

or *in extenso* if the states $|\Phi\,\phi\,\theta\rangle$ form an orthonormal set. In the SU(2) case a condition equivalent to (5.13) was incorrectly supposed to be satisfied in formulae (42*a*)–(51) of Rashid's paper (Rashid 1977a). Indeed, the SU(2) quasi-intelligent states form an overcomplete non-orthogonal set, but they can be normalised to unity, with the consequence that the diagonal elements of the analogue of (5.13) reduce to one. In going over to SU(1, 1) we do not expect to gain orthogonality. What is more, we have to be aware that matrix elements can behave as distributions. In the latter case it is suitable to define a new inner product as follows:

$$\langle\mu|\lambda\rangle_{\mathrm{i}} = \int\langle\mu|\lambda\rangle\,\mathrm{d}\lambda.\tag{5.14}$$

The same simplifications as with (5.13) are then obtained if

$$\langle\Phi\,\phi_2\,\theta|\Phi\,\phi_1\,\theta\rangle = \delta(\phi_2-\phi_1),\tag{5.15}$$

i.e. if the eigenfunctions are delta-normalisable.

In the special case $\theta_1=\theta_2=\theta$, and assuming that either (5.13) or (5.15) is satisfied, the expressions (5.11), (5.12) are thus reduced to:

$$\sinh^2(\theta+\theta^*)\langle\Phi\,\phi_2\,\theta)|\Delta K_1^2|\Phi\,\phi_1\,\theta\rangle = \sinh(\theta+\theta^*)\cosh\theta\cosh\theta^*\langle\Phi\,\phi_2\,\theta|K_3|\Phi\,\phi_1\,\theta\rangle,$$

$$\sinh^2(\theta+\theta^*)\langle\Phi\,\phi_2\,\theta|\Delta K_2^2|\Phi\,\phi_1\,\theta\rangle = \sinh(\theta+\theta^*)\sinh\theta\sinh\theta^*\langle\Phi\,\phi_2\,\theta|K_3|\Phi\,\phi_1\,\theta\rangle.$$

Clearly, if $\mathrm{Re}\,\theta\neq 0$ one has the equality

$$\langle\Delta K_1^2\rangle\langle\Delta K_2^2\rangle = \frac{\sinh\theta\cosh\theta\sinh\theta^*\cosh\theta^*}{\sinh^2(\theta+\theta^*)}\langle K_3\rangle^2,$$

or

$$\langle\Delta K_1^2\rangle\langle\Delta K_2^2\rangle = \frac{\sinh^2(2\,\mathrm{Re}\,\theta)-\sinh^2(2\,\mathrm{Im}\,\theta)}{4\sinh^2(2\,\mathrm{Re}\,\theta)}\langle K_3\rangle^2.\tag{5.16}$$

From the foregoing and taking into account that the right-hand side of (5.16) becomes equal to $\frac{1}{4}\langle K_3\rangle^2$ if $\mathrm{Im}(\theta)=0$, we can conclude that sufficient conditions for a quasi-intelligent state to be also intelligent are firstly that the state is normalised to unity as a function or as a distribution, and that secondly θ is a real number. If on the other hand $\theta = 0$, then K_3' is identical to K_2 and the corresponding eigenstates reduce to those given in (4.4). By taking the limit $\theta \to 0$ in the right-hand side of (5.16) one notices that these states can also be considered intelligent on condition of normalisation.

6. The normalisation constraint

In this section we investigate whether the states (3.9) are normalisable. To this end use is made of a result given by Barut and Phillips (1968). In the basis of eigenvectors of K_2 the authors derived the matrix elements of a K_3-rotation $e^{i\omega K_3}$:

$$\begin{aligned}
\langle\Phi\,\mu|e^{i\omega K_3}|\Phi\,\lambda\rangle
&= \frac{A_\lambda}{A_\mu}\frac{1}{2\pi}\exp[(\lambda-\mu)\pi/2]\bigg[\left(-\frac{b^2}{a^2}\right)^{\frac{1}{2}i(\lambda-\mu)}\frac{\Gamma(-\Phi-i\mu)\Gamma(i\mu-i\lambda)}{\Gamma(-\Phi-i\lambda)}\\
&\quad\times{}_2F_1(-\Phi+i\lambda,-\Phi-i\mu;1+i\lambda-i\mu;-b^2/a^2)\\
&\quad+\left(-\frac{b^2}{a^2}\right)^{\frac{1}{2}i(\mu-\lambda)}\frac{\Gamma(-\Phi+i\mu)\Gamma(i\lambda-i\mu)}{\Gamma(-\Phi+i\lambda)}\\
&\quad\times{}_2F_1(-\Phi-i\lambda,-\Phi+i\mu;1-i\lambda+i\mu;-b^2/a^2)\bigg],
\end{aligned} \tag{6.1}$$

where

$$a=\cos(\omega/2), \qquad b=\sin(\omega/2)$$

and the phase of $(-b)$ is chosen to be $e^{i\pi}$. With the help of (6.1) they also proved that the states $|\Phi\,\lambda\rangle$ are delta-normalised. Indeed, it can be shown by setting $\omega=0$ in (6.1) and by taking into account the formula

$$\lim_{\eta\to 0}[\eta^{ix}\Gamma(-ix)+\eta^{-ix}\Gamma(ix)]=2\pi\delta(x), \tag{6.2}$$

the prescription to surround the poles on the real x axis in the left-hand side of (6.2) being the principal value prescription, that:

$$\langle\Phi\,\mu|\Phi\,\lambda\rangle=\delta(\mu-\lambda). \tag{6.3}$$

From the operator identity (4.5) it is clear that

$$\langle\Phi\,\phi_2\,\theta|\Phi\,\phi_1\,\theta\rangle=\langle\Phi\,\phi_2|\exp[-(\theta+\theta^*)K_3]|\Phi\,\phi_1\rangle, \tag{6.4}$$

whereby $|\Phi\,\phi_i\rangle$ is the eigenstate of K_2 belonging to the real eigenvalue ϕ_i. Now the right-hand side of (6.4) is equivalent to the left-hand side of (6.1) if one substitutes:

$$\omega=i(\theta+\theta^*)=2i\,\mathrm{Re}\,\theta. \tag{6.5}$$

Although (6.1) is strictly valid only for real ω, the right-hand side of (6.1) can be analytically continued to pure imaginary ω. Nevertheless, it is easy to see that whatever the nature of ω is, the expression (6.1) becomes infinite for $\lambda=\mu$ due to the pole of $\Gamma(i\lambda-i\mu)$ or $\Gamma(i\mu-i\lambda)$, whereas for $\lambda\neq\mu$ and $\mathrm{Re}\,\theta\neq 0$ this expression is different from zero. This shows us that the eigenstates $|\Phi\ \phi\ \theta\rangle$ are not delta-normalisable. Only if Re $\theta = 0$, one obtains

$$\langle \Phi\ \phi_2\ \theta | \Phi\ \phi_1\ \theta \rangle = \delta(\phi_1 - \phi_2) \qquad \mathrm{Re}\ \theta = 0, \tag{6.6}$$

demonstrating that the K_2 eigenstates are intelligent states associated with the hyperbolic angular momentum algebra and the commutator $[K_2, K_1] = \mathrm{i}K_3$. It is obvious that the above results do not exclude the possible existence of a measure enabling normalisation of all the quasi-intelligent states.

Among the artificial methods to overcome the normalisation constraint problem, there is an almost trivial one. In analogy with the compact case, define a state

$$|\Phi\ \phi\ \theta^{\mathrm{c}}\rangle = |\Phi\ \phi - \theta^*\rangle$$

with the consequence that

$$\langle \Phi\ \phi_2\ \theta^{\mathrm{c}} | \Phi\ \phi_1\ \theta \rangle = \langle \Phi\ \phi_2 | \Phi\ \phi_1 \rangle = \delta(\phi_2 - \phi_1).$$

Defining further the expectation values of operators by

$$\langle \Phi\ \phi\ \theta^{\mathrm{c}} |\ \text{operator}\ | \Phi\ \phi\ \theta \rangle,$$

all the states $|\Phi\ \phi\ \theta\rangle$ with Im $\theta = 0$ are intelligent with respect to the newly defined inner product.

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